Casimir pistons with hybrid boundary conditions

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The Casimir effect giving rise to an attractive or repulsive force between the configuration boundaries that confine the massless scalar field is reexamined for one to three-dimensional pistons in this paper. Especially, we consider Casimir pistons with hybrid boundary conditions, where the boundary condition on the piston is Neumann and those on other surfaces are Dirichlet. We show that the Casimir force on the piston is always repulsive, in contrast with the same problem where the boundary conditions are Dirichlet on all surfaces.

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1. Introduction

Casimir predicted that an attractive force should act between two plane-parallel uncharged perfectly conducting plates in vacuum[1]. The force is due to the disturbance of the vacuum of the quantized electromagnetic field under the existence of the boundary conditions. This effect has now been verified by precise measurements[2] and has been applied to the fabrication of microelectromechanical systems(MEMS)[3]. Casimir energies and forces have been calculated theoretically in various different configurations including stratified media, rectangular cavities, wedge, sphere, cylinder, sphere(lens) above a disk and other[4]. The calculations indicated that the Casimir energy may change its sign depending not only on the boundary conditions but also on geometry and topology of the configuration. About the dependence of the Casimir force on the boundary conditions, we have known that the Casimir force is still attractive for two parallel infinitely permeable plates(associated to Neumann-Neumann boundary conditions), which is identical to Casimir's original result. However, the Casimir force between a perfectly conducting plate and an infinitely permeable one (associated to Dirichlet-Neumann boundary conditions) is repulsive[5]. Due to this peculiar aspect, the Casimir effect with hybrid boundary conditions(Dirichlet-Neumann) has been considered recent years[6]. As for the influence of geometry and topology of the configuration to the Casimir force, it has been claimed that the Casimir energy inside rectangular cavities can be either positive or negative depending on the ratio of the sides[7]. But the security of the conclusion needs to suspect at least for two reasons: First, the calculations ignore the divergent term associated with the boundary. With the term thrown out, the cut-off technique can get the same finite result as what is obtained by zeta function regularization which renormalizes the surface term to zero. But physically, such a term cannot be eliminated by a renormalization of the parameters of the theory[8]. Second, it does not take into account the nontrivial contribution to the vacuum energy from the outside region of the box. Actually, these problems exist in the calculation of Casimir energy of any single body. In contrast, there is no such problems for the case of two rigid bodies if one only interested in the force between them.

Two years ago, a modification of the rectangle-"Casimir piston"-was introduced to avoid the above two problems[9]. The two-dimensional Casimir piston consists of a single rectangle divided into two by a partition(the "piston"). The Casimir force on the piston is a well-defined finite force because the position of the piston is independent of the divergent terms in the internal vacuum energy and the external region. For a scalar field obeying Dirichlet boundary conditions on all surfaces, when the separation between the piston and one end of the cavity approaches infinity, the force on the piston is towards another end(the closed end), that is, the force is attractive. Later, the attractive Casimir force on the piston again obtained for a three-dimensional electromagnetic field with the perfect-conductor conditon[10] and for a three-dimensional scalar field with Dirichlet boundary conditions on all surfaces[11]. But attraction does not occur in all Casimir piston scenarios. In a recent paper[12], the Casimir piston in a cylinder for a weakly reflecting dielectric was considered and it was shown that the force could switch from attraction to repulsion with the plate separation increasing. Again, some examples of repulsive Casimir pistons have been simply discussed in a recent preprint[13].

On the other hand, it may be worth emphasizing that the Hurwitz zeta function $\zeta(\nu; s)$ is the direct zeta function

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associated with the hybrid boundary conditions, while the Riemann zeta function $\zeta(s)$ is the direct zeta function associated with the Dirichlet boundary condition. $\zeta(\nu;s)$ is a generalization of $\zeta(s)$ defined by

$$\zeta(\nu;s)=\sum_{n=0}^{\infty}\frac{1}{(n+\nu)^{s}}, \qquad (0<\nu\leq 1, \mathrm{Re}s>1) \tag{1}$$

It is obvious that $\zeta(s)=\zeta(1;s)$. For Re$s>0$, one has[14]

$$\Gamma(s)=(n+\nu)^{s}\int_{0}^{\infty}x^{s-1}e^{-(n+\nu)x}dx \tag{2}$$

Therefore,

$$\zeta(\nu;s)=\frac{1}{\Gamma(s)}\int_{0}^{\infty}\frac{x^{s-1}e^{-(\nu-1)x}}{e^{x}-1}dx \tag{3}$$

if the inversion of the order of summation and integration can be justified; and this is guaranteed by the absolute convergence if Re$s>1$. Now one can consider the integral

$$\zeta(\nu;s)=\frac{-e^{i\pi s}\Gamma(1-s)}{2\pi i}\int_{C}\frac{z^{s-1}e^{-\nu z}}{1-e^{-z}}dz \tag{4}$$

where the contour C starts at infinity on the positive real axis, encircles the origin once in the positive direction excluding the point $\pm 2i\pi,\pm 4i\pi,\cdots$, and returns to positive infinity. We can take C to consist of real axis from $+\infty$ to r $(0<r<2\pi)$, the circle $\mid z\mid=r$, and the real axis from r to $+\infty$. On making $r\to 0$, we have obtained Eq.(4). Eq.(4) provides the analytic continuation of $\zeta(\nu;s)$ over the whole plane, and $\zeta(\nu;s)$ is regular everywhere except for a simple pole at $s=1$ with residue 1. Expanding the loop to infinity, the residues are at $\pm 2mi\pi$; hence, if Re$s<0$, we have

$$\zeta(\nu;s)=\frac{2\Gamma(1-s)}{(2\pi)^{1-s}}\Big[\sin\frac{1}{2}\pi s\sum_{m=1}^{\infty}\frac{\cos 2m\pi\nu}{m^{1-s}}+\cos\frac{1}{2}\pi s\sum_{m=1}^{\infty}\frac{\sin 2m\pi\nu}{m^{1-s}}\Big] \tag{5}$$

In this paper, we consider Casimir pistons for a massless scalar field with hybrid boundary conditions. That is, on the surface where the piston lies, the boundary condition is Neumann, and the boundary conditions are Dirichlet on other surfaces. We discuss one to three dimensional pistons using generalized zeta function regularization technique. Due to the existence of hybrid boundary conditions, Hurwitz zeta functions and Epstein zeta functions emerge naturally. In one dimensional case, it is very easy to get the analytic result by regularizing Hurwitz zeta function directly. In two and three dimensional cases, we need to do the calculation numerically after the regularization. In all of the three cases, we show that the Casimir force on the piston with hybrid boundary conditions are repulsive. With the separation increasing, the force on the piston decreases rapidly.

2. One-dimensional piston

Consider a quantized scalar field constrained in the interval L on the real line(See Fig.1). There is a point in the interval where the piston lies. The piston divides the interval into two labeled A and B. The distance between the piston and the left point is a. The total energy of the vacuum for the system can be written as the sum of three terms:

$$E=E^{A}(a)+E^{B}(L-a)+E^{out} \tag{6}$$

where $E^{A}(a)$ and $E^{B}(L-a)$ are given by the results through cut-off technique, which consist of divergent terms and finite terms, where the finite terms are the same as what are obtained by zeta function regularization denoted $E_{R}^{A}(a)$ and $E_{R}^{B}(L-a)$. The divergent terms and the energy from the exterior region in the total energy are independent of the position of the piston[9], so the Casimir force on the piston is as follows:

$$F = -\frac{\partial}{\partial a}[E_R^A(a) + E_R^B(L-a)] \tag{7}$$

With Dirichlet boundary condition on one point and Neumann on the other point, the eigenfrequencies in interval A are

$$\omega_n = \frac{\left(n+\frac{1}{2}\right)\pi}{a}, \qquad n = 0, 1, 2, \cdots \tag{8}$$

So the vacuum energy is given by($\hbar = c = 1$ where c is the speed of light)

$$E(a) = \frac{1}{2}\sum_{n=0}^{\infty}(n+\frac{1}{2})\frac{\pi}{a} \tag{9}$$

Using zeta function regularization, we start with the function

$$\mathcal{E}(a;s) \equiv \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\left(n+\frac{1}{2}\right)\frac{1}{a}\right]^{-s} \tag{10}$$

which is defined for $\mathrm{Re}(s) > 1$. We will see in the following that its analytic continuation to the complex s-plane is well-defined at $s = -1$. So we can write the regularized Casimir energy as $E_R^A(a) = \mathcal{E}(a;-1)$.

Eq.(10) can be rewritten as

$$\mathcal{E}(a;s) \;=\; \frac{\pi a^s}{2}\zeta(\frac{1}{2};s) \tag{11}$$

Using Mellin transformation one can find

$$\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}(2^s-1)^{-1}\zeta(s) = \Gamma\left(\frac{1-s}{2}\right)\pi^{(s-1)/2}(2^{1-s}-1)^{-1}\zeta(1-s) \tag{12}$$

Taking $s = -1$, we get

$$\zeta(\frac{1}{2};-1) = \frac{1}{12\pi^2}\zeta(\frac{1}{2};2) = \frac{1}{24} \tag{13}$$

and so

$$E_R^A(a) = \mathcal{E}(a;-1) = \frac{\pi}{48a} \tag{14}$$

The corresponding expression for interval B is

$$E_R^B(L-a) = \frac{\pi}{48(L-a)} \tag{15}$$

Thus, taking the limit $L \to \infty$, we obtain the repulsive Casimir force on the piston

$$F = \frac{\pi}{48a^2} \tag{16}$$

The result is the same as what was obtained in [13] where an exponential cutoff technique was used.

3. Two-dimensional piston

As illustrated in Fig. 2, the rectangle is divided into two by the piston. The boundary condition on the piston is Neumann, and those on other surfaces are Dirichlet. Similar to one-dimensional case, the Casimir force acting on the piston is:

$$F = -\frac{\partial}{\partial a}\Big[E_R^A(a,b) + E_R^B(L-a,b)\Big] \tag{17}$$

The vacuum energy of area A is

$$E(a,b) = \frac{1}{2}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sqrt{\Big(m+\frac{1}{2}\Big)^2\Big(\frac{\pi}{a}\Big)^2 + \Big(\frac{n\pi}{b}\Big)^2} \tag{18}$$

In order to calculate the summation $E(a,b)$, we can consider the more general expression $\mathcal{E}(a,b;s)$ as follows

$$\begin{aligned}\mathcal{E}(a,b;s) &\equiv \frac{\pi}{2}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\Big[\Big(m+\frac{1}{2}\Big)^2\Big(\frac{1}{a}\Big)^2 + \Big(\frac{n}{b}\Big)^2\Big]^{-s/2} \\ &= \frac{\pi}{8}\sum_{m,n=-\infty}^{\infty}{}' \Big[\Big[\Big(\frac{m}{2a}\Big)^2 + \Big(\frac{n}{b}\Big)^2\Big]^{-s/2} - \Big[\Big(\frac{m}{a}\Big)^2 + \Big(\frac{n}{b}\Big)^2\Big]^{-s/2}\Big] - \frac{\pi a^s}{4}\zeta(\frac{1}{2};s) \\ &= \frac{\pi}{8}\Big[Z_2\Big(\frac{1}{2a},\frac{1}{b};s\Big) - Z_2\Big(\frac{1}{a},\frac{1}{b};s\Big)\Big] - \frac{\pi a^s}{4}\zeta(\frac{1}{2};s)\end{aligned} \tag{19}$$

Where $Z_p(a_1,\cdots,a_p;s)$ is Epstein zeta function which is defined as $Z_p(a_1,\cdots,a_p;s) \equiv \sum_{n_1,\cdots,n_p=-\infty}^{\infty}{}' \Big[(n_1a_1)^2+\cdots+(n_pa_p)^2\Big]^{-\frac{s}{2}}$ and the prime means that the term $n_1 = n_2 = \cdots = n_p = 0$ has to be excluded. Applying the reflection formulae

$$(a_1\cdots a_p)\Gamma\Big(\frac{s}{2}\Big)\pi^{-s/2}Z_p(a_1\cdots a_p;s) = \Gamma\Big(\frac{p-s}{2}\Big)\pi^{(s-p)/2}Z_p(\frac{1}{a_1}\cdots\frac{1}{a_p};p-s) \tag{20}$$

and taking $s=-1$, we get

$$E_R^A(a,b) = \mathcal{E}(a,b;-1) = -\frac{ab}{32\pi}\Big[2Z_2(2a,b;3) - Z_2(a,b;3)\Big] - \frac{\pi}{96a} \tag{21}$$

When $a > b$, we can reexpress the Epstein zeta function as[15]

$$Z_2(a,b;3) = \frac{2\pi^2}{3a^2b} + \frac{16\pi}{ab^2}\sum_{m,n=1}^{\infty}\frac{n}{m}K_1\Big(2\pi mn\frac{a}{b}\Big) + \frac{2\zeta(3)}{b^3} \tag{22}$$

Where $K_n(z)$ is modified Bessel function . Substituting Eq.(22) and the corresponding expression for $Z_2(2a,b;3)$ into Eq. (21), we get

$$E_R^A(a,b) = -\frac{\zeta(3)a}{16\pi b^2} - \frac{1}{2b}\sum_{m,n=1}^{\infty}\frac{n}{m}\Big[K_1\Big(4\pi mn\frac{a}{b}\Big) - K_1\Big(2\pi mn\frac{a}{b}\Big)\Big] \tag{23}$$

Inserting Eq. (23) and the corresponding expression for $E_R^B(L-a,b)$ into Eq. (17) and taking $L\to\infty$, we obtain the following result for the Casimir force on the piston

$$\lim_{L\to\infty} F = \frac{\pi}{b^2}\sum_{m,n=1}^{\infty} n^2\Big[2K_1'\Big(4\pi mn\frac{a}{b}\Big) - K_1'\Big(2\pi mn\frac{a}{b}\Big)\Big] \tag{24}$$

Where $K_1'(z) = dK_1(z)/dz$. The numerical calculation tells us that the force is positive and decreases rapidly with the ratio a/b increasing.

In the case that $a < b$, Eq. (22) changes to

$$Z_2(a,b;3) = \frac{2\pi^2}{3b^2 a} + \frac{16\pi}{ba^2}\sum_{m,n=1}^{\infty}\frac{n}{m}K_1\Big(2\pi mn\frac{b}{a}\Big) + \frac{2\zeta(3)}{a^3} \tag{25}$$

and the resulted force on the piston is

$$\lim_{L\to\infty} F = \frac{3b\zeta(3)}{32\pi a^3} - \frac{\pi}{96a^2} - \frac{\zeta(3)}{16\pi b^2} + \frac{\pi b}{4a^3}\sum_{m,n=1}^{\infty} n^2\Big[K_0\Big(\pi mn\frac{b}{a}\Big) - 4K_0\Big(2\pi mn\frac{b}{a}\Big)\Big] \tag{26}$$

It can also be shown that the force is positive and decreases with the value a/b increasing. Furthermore, one can know from the numerical computation that Eq. (24) and Eq. (26) are connected that they have the same value of the force when $a = b$.

4. Three-dimensional piston

Similarly, the results of two-dimensional pistons can be extended to those of three-dimensional pistons. The three-dimensional piston is depicted in Fig. 3, where again the boundary condition on the piston is Neumann and those on other surfacee are Dirichlet. For simplicity, we take the base as a square. The vacuum energy in cavity A is

$$E(a,b,b) = \frac{1}{2}\sum_{m=0}^{\infty}\sum_{n_1,n_2=1}^{\infty}\sqrt{\Big(m+\frac{1}{2}\Big)^2\Big(\frac{\pi}{a}\Big)^2 + \Big(\frac{n_1\pi}{b}\Big)^2 + \Big(\frac{n_2\pi}{b}\Big)^2} \tag{27}$$

When $a > b$, we get the regularized vacuum energy in cavity A as

$$E_R^A(a,b,b) = -\frac{a\beta(2)}{48b^2} + \frac{\zeta(3)a}{16\pi b^2} + \frac{1}{2b}\sum_{m,n_1,n_2=1}^{\infty}\frac{\sqrt{n_1^2+n_2^2}}{m}\Big[K_1\Big(2\pi m\sqrt{n_1^2+n_2^2}\frac{a}{b}\Big) - K_1\Big(4\pi m\sqrt{n_1^2+n_2^2}\frac{a}{b}\Big)\Big] \tag{28}$$

where $\beta(2)$ is a Dirichlet series defined as $\beta(s) \equiv \sum_{n=0}^{\infty}(-1)^n(2n+1)^{-s}$ which comes from the relation $Z_2(1,1;s) = 4\zeta(s)\beta(s)$[16] during the regularization. Substituting Eq.(28) and the corresponding expression for the regularized vacuum energy in cavity B into the following expression for Casimir force on the piston

$$F = -\frac{\partial}{\partial a}\Big[E_R^A(a,b,b) + E_R^B(L-a,b,b)\Big] \tag{29}$$

and taking $L \to \infty$, we obtain the force on the piston as

$$\lim_{L\to\infty} F = \frac{\pi}{b^2}\sum_{m,n_1,n_2=1}^{\infty}(n_1^2+n_2^2)\Big[2K_1'\Big(4\pi m\sqrt{n_1^2+n_2^2}\frac{a}{b}\Big) - K_1'\Big(2\pi m\sqrt{n_1^2+n_2^2}\frac{a}{b}\Big)\Big] \tag{30}$$

The force is positive from the result of numerical calculation and it approaches zero with the ratio of a/b approaching infinity.

In the case that $a < b$, the regularized vacuum energy in cavity A can be reexpressed as

$$\begin{aligned} E_R^A(a,b,b) &= \frac{7\pi^2 b^2}{11720a^3} - \frac{3\zeta(3)b}{64\pi a^2} + \frac{\pi}{192a} + \frac{1}{4a}\sum_{m,n=1}^{\infty}\frac{m}{n}\Big[K_1(\pi mn\frac{b}{a}) - 2K_1(2\pi mn\frac{b}{a})\Big] \\ &+ \frac{b^{1/2}}{2a^{3/2}}\sum_{m,n=1}^{\infty}\Big(\frac{m}{n}\Big)^{3/2}\Big[K_{3/2}(2\pi mn\frac{b}{a}) - \frac{\sqrt{2}}{4}K_{3/2}(2\pi mn\frac{b}{a})\Big] \\ &+ \frac{b^{1/2}}{2a^{3/2}}\sum_{m,n_1,n_2=1}^{\infty}\Big(\frac{m}{\sqrt{n_1^2+n_2^2}}\Big)^{3/2}\Big[K_{3/2}(2\pi m\sqrt{n_1^2+n_2^2}\frac{b}{a}) - \frac{\sqrt{2}}{4}K_{3/2}(2\pi m\sqrt{n_1^2+n_2^2}\frac{b}{a})\Big] \end{aligned} \tag{31}$$

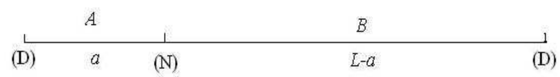

FIG. 1: Casimir piston in one dimensions.

Then the force on the piston is

$$
\begin{aligned}
\lim_{L\to\infty} F &= \frac{7\pi^2 b^2}{3840a^4} - \frac{3\zeta(3)b}{32\pi a^3} + \frac{\pi}{192a^2} - \frac{\beta(2)}{48b^2} + \frac{\zeta(3)}{16\pi b^2} \\
&- \frac{\pi b}{4a^3}\sum_{m,n=1}^{\infty} m^2\Big[K_0(\pi mn\frac{b}{a}) - 4K_0(2\pi mn\frac{b}{a})\Big] - \frac{\pi b^{3/2}}{a^{7/2}}\sum_{m,n=1}^{\infty}\frac{m^{5/2}}{n^{1/2}}\Big[K_{1/2}(2\pi mn\frac{b}{a}) - \frac{\sqrt{2}}{8}K_{1/2}(\pi mn\frac{b}{a})\Big] \\
&- \frac{\pi b^{3/2}}{a^{7/2}}\sum_{m,n_1,n_2=1}^{\infty}\frac{m^{5/2}}{(n_1^2+n_2^2)^{1/4}}\Big[K_{1/2}(2\pi m\sqrt{n_1^2+n_2^2}\frac{b}{a}) - \frac{\sqrt{2}}{8}K_{1/2}(\pi m\sqrt{n_1^2+n_2^2}\frac{b}{a})\Big] \qquad (32)
\end{aligned}
$$

The force is again positive and decreases with the ratio a/b increasing (see Fig.4).

For the special case that $a = b$, which means cavity A is a cube, we find from both Eq. (30) and Eq. (32) that the force on the piston is (in unit $\hbar c$) $F = \frac{0.00041244}{b^2}$.

5. Conclusion

We discuss one to three-dimensional Casimir pistons for a massless scalar field with hybrid boundary conditions, where the boundary condition on the piston is Neumann and those on other surfaces are Dirichlet. We find the forces on the pistons are always repulsive, in contrast with the same problem where the boundary conditions are Dirichlet on all surfaces.

The problem of hybrid boundary conditions we study here is in analogue with the problem in electromagnetic field that the piston is an infinitely permeable plate and the other sides of the cavity are perfectly conducting ones or the opposite case that the piston is a perfectly conducting plate and the other sides are infinitely permeable ones. This problem may be connected with the study of dynamical Casimir effect and may be applied to the fabrication of MEMS, which needs further investigation.

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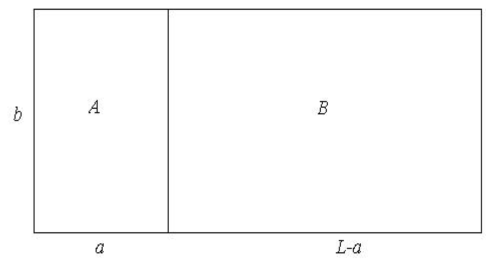

FIG. 2: Casimir piston in two dimensions.

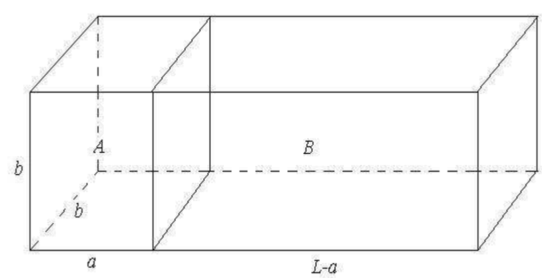

FIG. 3: Casimir piston in three dimensions.

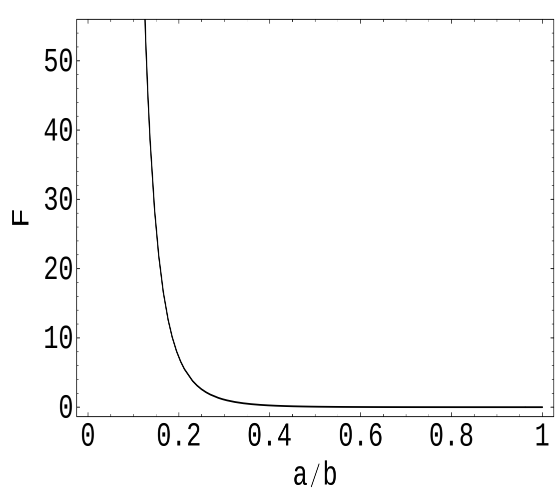

FIG. 4: Casimir force F (in units $\hbar c/b^2$) on a three-dimensional piston versus a/b where a is the plate separation and b is the length of the sides of the square base.

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